Exploding dice

Expected value for dice that can re-roll

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What are exploding dice?

Dice that re-roll (based on certain value) and add on the the initial value

Example: {1, 2, 3, 4, 5, 6+}

- Roll: 3 → 3
- Roll: 6+2 → 8
- Roll: 6+6+5 → 17

How can we compute the average (aka. expected value)?

Other examples of exploding dice

(many by designer Eric Lang)

- Bloodborne card game, green: {0, 0, 1, 1+, 1+, 2}
- Bloodborne card game, yellow: {0, 1, 1, 1+, 2+, 3}
- Bloodborne card game, red: {0, 1+, 2, 2+, 3, 4}
- Arcadia Quest melee: {1, 1, 1, 0, 0, 1+}
- Arcadia Quest ranged: {0, 0, 0, 1, 1, 1+}
- Arcadia Quest defense: {0, 0, 0, 0, 1, 1+}

Expected value

$$EV = \text{value}_1 \cdot \text{probability}_1 + \text{value}_2 \cdot \text{probability}_2 + \text{value}_3 \cdot \text{probability}_3 + \ldots$$

Expected value of six-sided dice: {1, 2, 3, 4, 5, 6}

$$\left(1 \cdot \frac{1}{6}\right) + \left(2 \cdot \frac{1}{6}\right) + \left(3 \cdot \frac{1}{6}\right) + \left(4 \cdot \frac{1}{6}\right) + \left(5 \cdot \frac{1}{6}\right) + \left(6 \cdot \frac{1}{6}\right)$$

$$= (1 + 2 + 3 + 4 + 5 + 6) \cdot \frac{1}{6}$$

$$= \frac{21}{6} = 3.5$$

Previous analysis

via Eric T Dobbs (2009)

$$\left(1 \cdot \frac{1}{6}\right) + \left(2 \cdot \frac{1}{6}\right) + \left(3 \cdot \frac{1}{6}\right) + \left(4 \cdot \frac{1}{6}\right) + \left(5 \cdot \frac{1}{6}\right) +$$

$$\left(7 \cdot \frac{1}{36}\right) + \left(8 \cdot \frac{1}{36}\right) + \left(9 \cdot \frac{1}{36}\right) + \left(10 \cdot \frac{1}{36}\right) + \left(11 \cdot \frac{1}{36}\right) +$$

$$\left(13 \cdot \frac{1}{216}\right) + \dots$$

$$= \sum_{N=0}^{\infty} \frac{2.5 + 5 \cdot N}{6^N}$$

$$= 4.2$$

Previous analysis

via Eric T Dobbs (2009)

Implications: For any N-sided die numbered 1 to N with all sides equally likely, the exploding modifier will modify the die's expected value by a factor of N/(N-1)

Example: {1, 2, 3, 4, 5, 6+}

- Normal expected value: 3.5
- Exploding expected value: 3.5 * 6/5 = 4.2

Limitations:

- Only works with the last number exploding
- Assumes dice are 1 to N

Monte Carlo simulation
(via Python)

```
import random
numTrials = 10**7
diceSides = [1, 2, 3, 4, 5, 6]
sum = 0
trials = 0
while trials < numTrials:
    dice = random.choice(diceSides)
    sum += dice
    # move on to next trial if non-six, otherwise, stay on current trial
    if dice != 6:
        trials += 1
print("average: ", sum/numTrials)
```

Trial #1: 4.2010773

Trial #2: 4.2001453

Trial #3: 4.1989661

Trial #4: 4.1999724

Trial #5: 4.2001959

Trial #6: 4.199956458 (numTrials=10^9)

Trial #7: 4.1999714361 (numTrials=10^10)

Trial #8: 4.200002724505 (numTrials=10^12)

Different analysis

dice: {1, 2, 3, 4, 5, 6+}

- First roll happens once, average of 3.5
- Second roll happens 1/6 of the time, average of 3.5
- Third roll happens 1/36 of the time, average of 3.5
- Fourth roll happens 1/216 of the time, average of 3.5 ...

New analysis

dice: {1, 2, 3, 4, 5, 6+}

- First roll happens once, average of 3.5
- Second roll happens 1/6 of the time, average of 3.5
- Third roll happens 1/36 of the time, average of 3.5
- Fourth roll happens 1/216 of the time, average of 3.5 ...

$$(3.5 \cdot 1) + \left(3.5 \cdot \frac{1}{6}\right) + \left(3.5 \cdot \frac{1}{36}\right) + \left(3.5 \cdot \frac{1}{216}\right) + \ldots$$

$$= 3.5\left(1 + \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \ldots\right)$$

$$= 3.5 \sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n$$

Geometric series

$$S = \sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n$$

$$S = 1 + \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \dots$$

$$\frac{1}{6}S = \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \dots$$

$$S - \frac{1}{6}S = 1$$

$$S = \frac{6}{5}$$

General:

$$S = \sum_{n=0}^{\infty} r^n, |r| < 1$$

$$S = \frac{1}{1 - r}$$

Different analysis

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$$(3.5 \cdot 1) + \left(3.5 \cdot \frac{1}{6}\right) + \left(3.5 \cdot \frac{1}{36}\right) + \left(3.5 \cdot \frac{1}{216}\right) + \dots$$

$$= 3.5\left(1 + \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \dots\right)$$

$$= 3.5 \sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n = 3.5 \cdot \frac{6}{5} = 4.2$$

Expected value for exploding dice

$$
\begin{aligned}
EV &= \text{average} \cdot \sum_{n=0}^{\infty} P(\text{reroll})^n \\
&= \frac{\text{sum}}{\text{\# sides}} \cdot \frac{1}{1 - P(\text{reroll})} \\
&= \frac{\text{sum}}{\text{\# sides}} \cdot \frac{1}{1 - \frac{\text{\# rerolls}}{\text{\# sides}}} \\
&= \frac{\text{sum}}{\text{\# sides} - \text{\# rerolls}} = \frac{\text{sum}}{\text{\# normal sides}}
\end{aligned}
$$

Compare to previous analysis

$$EV_{\text{Dobbs}} = \text{average} \cdot \frac{n}{n-1}$$

- $n = 4 \rightarrow 2.5 \cdot \frac{4}{3} = 3.333...$
- $n = 6 \rightarrow 3.5 \cdot \frac{6}{5} = 4.2$
- $n = 10 \rightarrow 5.5 \cdot \frac{10}{9} = 6.111...$
- $n = 20 \rightarrow 10.5 \cdot \frac{20}{19} \approx 11.05$
- $n = 100 \rightarrow 50.5 \cdot \frac{100}{99} = 51.01...$

$$EV_{\text{Hlas}} = \frac{\text{sum}}{\text{\# normal sides}}$$

- $n = 4 \rightarrow \frac{10}{3} = 3.333...$
- $n = 6 \rightarrow \frac{21}{5} = 4.2$
- $n = 10 \rightarrow \frac{55}{9} = 6.111...$
- $n = 20 \rightarrow \frac{210}{9} \approx 11.05$
- $n = 100 \rightarrow \frac{5050}{99} = 51.01...$

Compare to previous analysis

$$EV_{\text{Dobbs}} = \text{average} \cdot \frac{n}{n-1}$$

$$EV_{\text{Hlas}} = \frac{\text{sum}}{\text{\# normal sides}} = \frac{\text{sum}}{\text{\# sides} - \text{\# rerolls}}$$

$$EV_{\text{Dobbs}} = \text{average} \cdot \frac{n}{n-1} = \frac{\text{sum}}{n} \cdot \frac{n}{n-1} = \frac{\text{sum}}{n-1} = EV_{\text{Hlas}}$$

Other examples of exploding dice
(many by designer Eric Lang)

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- Arcadia Quest ranged: {0, 0, 0, 1, 1, 1+}
- Arcadia Quest defense: {0, 0, 0, 0, 1, 1+}

Bloodbourne card game

green {0, 0, 1, 1+, 1+, 2} : $$EV = \frac{0+0+1+1+1+2}{4} = \frac{5}{4} = 1.2$$

yellow {0, 1, 1, 1+, 2+, 3} : $$EV = \frac{0+1+1+1+2+3}{4} = \frac{8}{4} = 2$$

red {0, 1+, 2, 2+, 3, 4} : $$EV = \frac{0+1+2+2+3+4}{4} = \frac{12}{4} = 3$$

Arcadia Quest game

melee {1, 1, 1, 0, 0, 1+} : $$EV = \frac{1+1+1+0+0+1}{5} = \frac{4}{5} = 0.8$$

ranged {0, 0, 0, 1, 1, 1+} : $$EV = \frac{0+0+0+1+1+1}{5} = \frac{3}{5} = 0.6$$

defense {0, 0, 0, 0, 1, 1+} : $$EV = \frac{0+0+0+0+1+1}{5} = \frac{2}{5} = 0.2$$

GameTek examples

Uncharted Seas {0, 0, 0, 1, 1, 2+} : $$EV = \frac{0+0+0+1+1+2}{5} = \frac{4}{5} = 0.8$$

prototype {0, 0, 0, 1, 1+, 2+} : $$EV = \frac{0+0+0+1+1+2}{4} = \frac{4}{4} = 1$$

GameTek #1: Power laws
https://www.getrevue.co/profile/gengelstein/issues/gametek-issue-1-power-laws-647946

GameTek #14: Small chances, big problems
https://www.getrevue.co/profile/gengelstein/issues/gametek-14-small-chances-big-problems-1053991

Alternate analysis

$$E(x) = \mu + \frac{1}{6} \cdot E(x)$$

$$\frac{5}{6}E(x) = \mu$$

$$E(x) = \frac{6}{5} \cdot \mu$$

$$E(x) = \frac{1}{1 - P(\text{reroll})} \cdot \mu$$

Comparison #1

{0, 0, 0, 0, 0, 6+} vs. {1, 1, 1, 1, 1, 1+}

Left {0, 0, 0, 0, 0, 6+} : $$EV = \frac{0+0+0+0+0+6}{5} = \frac{6}{5} = 1.2$$

Right {1, 1, 1, 1, 1, 1+} : $$EV = \frac{1+1+1+1+1+1}{5} = \frac{6}{5} = 1.2$$

experimental ~= 1.2

Comparison #2

{0, 0, 0, 0, 0, 6+} vs. {0+, 0, 0, 0, 0, 6}

Left {0, 0, 0, 0, 0, 6+} :

$$EV = \frac{0+0+0+0+0+6}{5} = \frac{6}{5} = 1.2$$

Right {0+, 0, 0, 0, 0, 6} :

$$EV = \frac{0+0+0+0+0+6}{5} = \frac{6}{5} = 1.2$$

experimental ~= 1.2

Comparison #3

{0, 100+} vs. {0+, 100}

Left {0, 100+} : $$EV = \frac{0 + 100}{1} = \frac{100}{1} = 100$$

Right {0+, 100} : $$EV = \frac{0 + 100}{1} = \frac{100}{1} = 100$$

experimental ~= 100

Comparison #4

{0, 50, 100+} vs. { 0, 50+, 100} vs. {0+, 50, 100}

Left {0, 50, 100+} : $$EV = \frac{0 + 50 + 100}{2} = \frac{150}{2} = 75$$

Middle {0, 50+, 100} : $$EV = \frac{0 + 50 + 100}{32} = \frac{150}{2} = 75$$

Right {0+, 50, 100} : $$EV = \frac{0 + 50 + 100}{2} = \frac{150}{2} = 75$$

experimental ~= 75

Sources

- Mathematical analysis of exploding dice
  https://eric22222.wordpress.com/2009/03/22/a-mathematical-analysis-of-exploding-dice/
- Exploding dice
  http://polymathprogrammer.com/2009/08/06/exploding-dice/
- RolePlayingGameGeek > Exploding dice
  https://rpggeek.com/rpgmechanic/2129/exploding-dice-die-may-explode-second-roll
- StackExchange > Odds of rolling exploding dice
  https://math.stackexchange.com/questions/1087444/odds-of-rolling-exploding-dice
- Games featuring exploding dice
  https://www.reddit.com/r/boardgames/comments/2wwwuh/which_games_feature_exploding_dice/
- The math behind exploding dice rolls
  https://www.analyticscheck.net/posts/exploding-dice

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Thanks!