Exploding dice Expected value for dice that can re-roll

Prof. Chris Hlas, University of Wisconsin-Eau Claire

What are exploding dice?

Dice that re-roll (based on certain value) and add on the the initial value

Example: {1, 2, 3, 4, 5, 6+}

- Roll: $3 \rightarrow 3$
- Roll: 6+2 → 8
- Roll: 6+6+5 → 17

How can we compute the average (aka. expected value)?

Other examples of exploding dice (many by designer Eric Lang)

- Bloodborne card game, green: {0, 0, 1, 1+, 1+, 2}
- Bloodborne card game, yellow: {0, 1, 1, 1+, 2+, 3}
- Bloodborne card game, red: {0, 1+, 2, 2+, 3, 4}
- Arcadia Quest melee: {1, 1, 1, 0, 0, 1+}
- Arcadia Quest ranged: {0, 0, 0, 1, 1, 1+}
- Arcadia Quest defense: {0, 0, 0, 0, 1, 1+}

Expected value

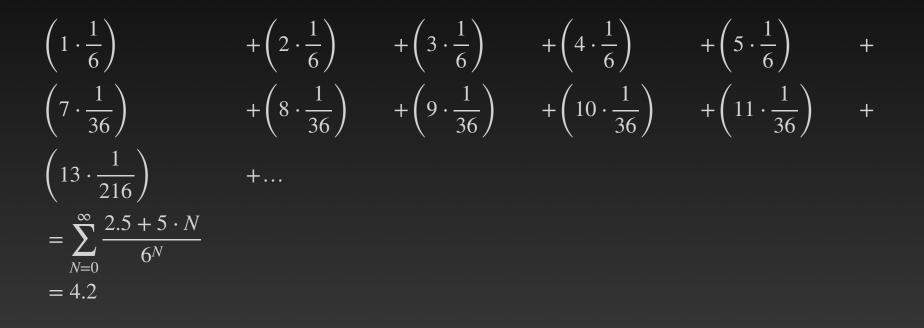
 $EV = value_1 \cdot probability_1 + value_2 \cdot probability_2 + value_3 \cdot probability_3 + \dots$

Expected value of six-sided dice: {1, 2, 3, 4, 5, 6}

$$\left(1 \cdot \frac{1}{6}\right) + \left(2 \cdot \frac{1}{6}\right) + \left(3 \cdot \frac{1}{6}\right) + \left(4 \cdot \frac{1}{6}\right) + \left(5 \cdot \frac{1}{6}\right) + \left(6 \cdot \frac{1}{6}\right)$$

 $= (1 + 2 + 3 + 4 + 5 + 6) \cdot \frac{1}{6}$
 $= \frac{21}{6} = 3.5$

Previous analysis via <u>Eric T Dobbs</u> (2009)



Previous analysis via <u>Eric T Dobbs</u> (2009)

Implications: For any N-sided die numbered 1 to N with all sides equally likely, the exploding modifier will modify the die's expected value by a factor of N/(N-1)

Example: {1, 2, 3, 4, 5, 6+}

- Normal expected value: 3.5
- Exploding expected value: 3.5 * 6/5 = 4.2

Limitations:

- Only works with the last number exploding
- Assumes dice are 1 to N

Monte Carlo simulation (via Python)

import random
numTrials = 10**7
diceSides = [1, 2, 3, 4, 5, 6]
sum = 0
trials = 0
while trials < numTrials:
 dice = random.choice(diceSides)
 sum += dice
 # move on to next trial if non-six, otherwise, stay on current trial
 if dice != 6:
 trials += 1
print("average: ", sum/numTrials)</pre>

Trial #1: 4.2010773 Trial #2: 4.2001453 Trial #3: 4.1989661 Trial #4: 4.1999724 Trial #5: 4.2001959 Trial #6: 4.199956458 (numTrials=10^9) Trial #7: 4.1999714361 (numTrials=10^10) Trial #8: 4.200002724505 (numTrials=10^12)

Different analysis dice: {1, 2, 3, 4, 5, 6+}

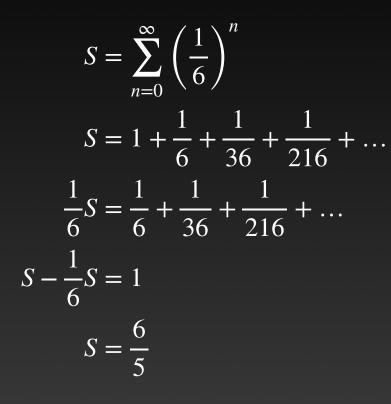
- First roll happens once, average of 3.5
- Second roll happens 1/6 of the time, average of 3.5
- Third roll happens 1/36 of the time, average of 3.5
- Fourth roll happens 1/216 of the time, average of 3.5 ...

New analysis dice: {1, 2, 3, 4, 5, 6+}

- First roll happens once, average of 3.5
- Second roll happens 1/6 of the time, average of 3.5
- Third roll happens 1/36 of the time, average of 3.5
- Fourth roll happens 1/216 of the time, average of 3.5 ...

$$(3.5 \cdot 1) + (3.5 \cdot \frac{1}{6}) + (3.5 \cdot \frac{1}{36}) + (3.5 \cdot \frac{1}{216}) + \dots$$
$$= 3.5 \left(1 + \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \dots\right)$$
$$= 3.5 \sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^{n}$$

Geometric series



General:

$$S = \sum_{n=0}^{\infty} r^{n}, |r| < 1$$

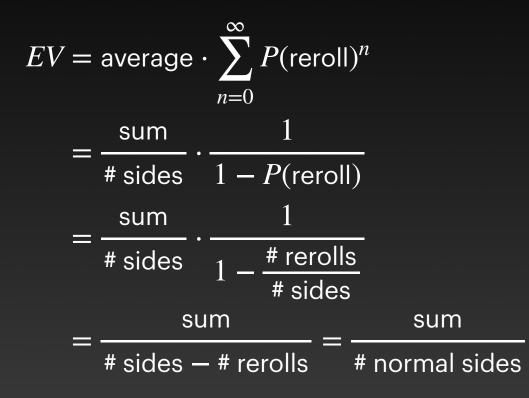
$$S = \frac{1}{1-r}$$

Different analysis dice: {1, 2, 3, 4, 5, 6+}

- First roll happens once, average of 3.5
- Second roll happens 1/6 of the time, average of 3.5
- Third roll happens 1/36 of the time, average of 3.5
- Fourth roll happens 1/216 of the time, average of 3.5 ...

$$(3.5 \cdot 1) + (3.5 \cdot \frac{1}{6}) + (3.5 \cdot \frac{1}{36}) + (3.5 \cdot \frac{1}{216}) + \dots$$
$$= 3.5 \left(1 + \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \dots\right)$$
$$= 3.5 \sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n = 3.5 \cdot \frac{6}{5} = 4.2$$

Expected value for exploding dice



Compare to previous analysis

S

• • •

$$EV_{\text{Dobbs}} = \text{average} \cdot \frac{n}{n-1}$$

$$EV_{\text{Hlas}} = \frac{\text{sum}}{\text{\# normal sides}}$$

$$n = 4 \to 2.5 \cdot \frac{4}{3} = 3.333...$$

$$n = 4 \to \frac{10}{3} = 3.333...$$

$$n = 4 \to \frac{10}{3} = 3.333...$$

$$n = 6 \to 3.5 \cdot \frac{6}{5} = 4.2$$

$$n = 10 \to 5.5 \cdot \frac{10}{9} = 6.111...$$

$$n = 10 \to \frac{55}{9} = 6.111...$$

$$n = 20 \to 10.5 \cdot \frac{20}{19} \approx 11.05$$

$$n = 100 \to 50.5 \cdot \frac{100}{99} = 51.01...$$

$$n = 100 \to \frac{5050}{99} = 51.01...$$

Compare to previous analysis

$$EV_{\text{Dobbs}} = \text{average} \cdot \frac{n}{n-1}$$
 $EV_{\text{Hlas}} = \frac{\text{sum}}{\text{\# normal sides}} = \frac{\text{sum}}{\text{\# sides} - \text{\# rerolls}}$

$$EV_{\text{Dobbs}} = \text{average} \cdot \frac{n}{n-1} = \frac{\text{sum}}{n} \cdot \frac{n}{n-1} = \frac{\text{sum}}{n-1} = EV_{\text{Hlas}}$$

Other examples of exploding dice (many by designer Eric Lang)

- Bloodborne card game, green: {0, 0, 1, 1+, 1+, 2}
- Bloodborne card game, yellow: {0, 1, 1, 1+, 2+, 3}
- Bloodborne card game, red: {0, 1+, 2, 2+, 3, 4}
- Arcadia Quest melee: {1, 1, 1, 0, 0, 1+}
- Arcadia Quest ranged: {0, 0, 0, 1, 1, 1+}
- Arcadia Quest defense: {0, 0, 0, 0, 1, 1+}

Bloodbourne card game

green {0, 0, 1, 1+, 1+, 2}:
$$EV = \frac{0+0+1+1+1+2}{4} = \frac{5}{4} = 1.2$$

yellow {0, 1, 1, 1+, 2+, 3}: $EV = \frac{0+1+1+1+2+3}{4} = \frac{8}{4} = 2$
red {0, 1+, 2, 2+, 3, 4}: $EV = \frac{0+1+2+2+3+4}{4} = \frac{12}{4} = 3$

Arcadia Quest game

melee {1, 1, 1, 0, 0, 1+}: $EV = \frac{1+1+1+0+0+1}{5} = \frac{4}{5} = 0.8$ ranged {0, 0, 0, 1, 1, 1+}: $EV = \frac{0+0+0+1+1+1}{5} = \frac{3}{5} = 0.6$ defense {0, 0, 0, 0, 1, 1+}: $EV = \frac{0+0+0+0+1+1}{5} = \frac{2}{5} = 0.2$

GameTek examples

Uncharted Seas {0, 0, 0, 1, 1, 2+}:
$$EV = \frac{0+0+0+1+1+2}{5} = \frac{4}{5} = 0.8$$

prototype {0, 0, 0, 1, 1+, 2+}: $EV = \frac{0+0+0+1+1+2}{4} = \frac{4}{4} = 1$

GameTek #1: Power laws <u>https://www.getrevue.co/profile/gengelstein/issues/gametek-issue-1-power-laws-647946</u>

GameTek #14: Small chances, big problems <u>https://www.getrevue.co/profile/gengelstein/issues/gametek-14-small-chances-big-problems-1053991</u>

Alternate analysis

$$E(x) = \mu + \frac{1}{6} \cdot E(x)$$

$$\frac{5}{6}E(x) = \mu$$

$$E(x) = \frac{6}{5} \cdot \mu$$

$$E(x) = \frac{1}{1 - P(\text{reroll})} \cdot \mu$$

Comparison #1 {0, 0, 0, 0, 0, 6+} vs. {1, 1, 1, 1, 1, 1+}

Left {0, 0, 0, 0, 0, 6+} :

Right {1, 1, 1, 1, 1, 1+} :

$$EV = \frac{0+0+0+0+0+6}{5} = \frac{6}{5} = 1.2$$
$$EV = \frac{1+1+1+1+1+1}{5} = \frac{6}{5} = 1.2$$

experimental ~= 1.2

Comparison #2 {0, 0, 0, 0, 0, 6+} vs. {0+, 0, 0, 0, 0, 0, 6}

Left {0, 0, 0, 0, 0, 6+} :

Right {0+, 0, 0, 0, 0, 6} :

$$EV = \frac{0+0+0+0+6}{5} = \frac{6}{5} = 1.2$$
$$EV = \frac{0+0+0+0+6}{5} = \frac{6}{5} = 1.2$$

experimental ~= 1.2

Comparison #3 {0, 100+} vs. {0+, 100}

Left {0, 100+}:
$$EV = \frac{0+100}{1} = \frac{100}{1} = 100$$

Right {0+, 100}: $EV = \frac{0+100}{1} = \frac{100}{1} = 100$

experimental ~= 100

Comparison #4 {0, 50, 100+} vs. { 0, 50+, 100} vs. {0+, 50, 100}

Left {0, 50, 100+}: $EV = \frac{0+50+100}{2} = \frac{150}{2} = 75$ Middle {0, 50+, 100}: $EV = \frac{0+50+100}{32} = \frac{150}{2} = 75$ Right {0+, 50, 100}: $EV = \frac{0+50+100}{2} = \frac{150}{2} = 75$

experimental ~= 75

Sources

- Mathematical analysis of exploding dice
 https://eric22222.wordpress.com/2009/03/22/a-mathematical-analysis-of-exploding-dice/
- Exploding dice <u>http://polymathprogrammer.com/2009/08/06/exploding-dice/</u>
- RolePlayingGameGeek > Exploding dice https://rpggeek.com/rpgmechanic/2129/exploding-dice-die-may-explode-second-roll
- StackExchange > Odds of rolling exploding dice <u>https://math.stackexchange.com/questions/1087444/odds-of-rolling-exploding-dice</u>
- Games featuring exploding dice
 https://www.reddit.com/r/boardgames/comments/2wwwuh/which_games_feature_exploding_dice/
- The math behind exploding dice rolls
 <u>https://www.analyticscheck.net/posts/exploding-dice</u>



Thanks!